CCRT: Categorical and Combinatorial Representation Theory.

From combinatorics of universal problems
to usual applications.

G.H.E. Duchamp
Collaboration at various stages of the work
and in the framework of the Project
Evolution Equations in Combinatorics and Physics :
Karol A. Penson, Darij Grinberg, Hoang Ngoc Minh, C. Lavault,
C. Tollu, N. Behr, V. Dinh, C. Bui,
Q.H. Ngô, N. Gargava, S. Goodenough, J.-Y. Enjalbert.

CIP seminar,
Friday conversations:
For this seminar, please have a look at Slide CCRT[n] & ff.

Outline

Goal of this series of talks

The goal of these talks is threefold

1. Category theory aimed at "free formulas" and their combinatorics
2. How to construct free objects
   1. w.r.t. a functor with - at least - two combinatorial applications:
      1. the two routes to reach the free algebra
      2. alphabets interpolating between commutative and non commutative worlds
   2. without functor: sums, tensor and free products
   3. w.r.t. a diagram: limits
3. Representation theory: Categories of modules, semi-simplicity, isomorphism classes i.e. the framework of Kronecker coefficients.
4. MRS factorisation: A local system of coordinates for Hausdorff groups.

Disclaimer. – The contents of these notes are by no means intended to be a complete theory. Rather, they outline the start of a program of work which has still not been carried out.

As an appetizer: a plea in favor of deep learning.

Of course time is short and pressing, but let me first begin by a compendium about the value of training to form the researcher (source [19], *Problems in Algebraic Number Theory*, preface to the first edition).

It is said that Ramanujan taught himself mathematics by systematically working through 6000 problems[a] *of Carr's Synopsis of Elementary Results in Pure and Applied Mathematics.*
Freeman Dyson in his DISTURBING THE UNIVERSE *describes the mathematical days of his youth when he spent his summer months working through hundreds of problems in differential equations.*
If we look back at our own mathematical development, we can certify that problem solving plays an important role in the training of the research mind.
In fact, it would not be an exaggeration to say that the ability to do research is essentially the art of asking the "right" questions. I suppose Pólya summarized this in his famous dictum: "IF YOU CAN'T SOLVE A PROBLEM, THEN THERE IS AN EASIER PROBLEM YOU CAN SOLVE - FIND IT!"

---

[a] Actually, Carr's Synopsis is not a problem book. It is a collection of theorems used by students to prepare themselves for the Cambridge Tripos. Ramanujan made it famous by using it as a problem book.

CCRT[22] MRS and the outer world III.

$\mathfrak{M}$-adic topologies/holomorphic functional calculus/extensions.

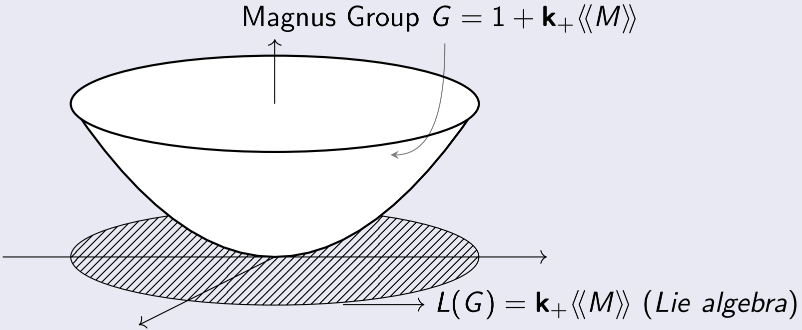
1. In the preceding weeks, we have considered the MRS factorization which is one of our precious jewels.

$$\mathcal{D}_X := \sum_{w \in X^*} w \otimes w = \sum_{w \in X^*} S_w \otimes P_w = \prod_{l \in \mathcal{L}ynX}^{\searrow} \exp(S_l \otimes P_l) \tag{1}$$

2. As was said many many times, it is an equality between **infinite sums and products** so we need to examine in detail the **limiting processes** involved there.
3. It can be considered:
    1. As an identity within the Magnus group of double series.
    2. As an identity between operators.
4. The equality $B = D$ can be also considered as a tool to factorize characters.

Identity $B = C = D$ within the Magnus group of double series.

5. Here the monoid will be taken firstly as $M = X^* \otimes X^*$, but we want to serve the generality in order to taylor a more general tool.



Which monoids are admissible for producing Magnus Groups ?

6. We recall that, a set E being given, $\mathbf{k}^E$ is the set of all functions $f \in E \to \mathbf{k}$,
   1. $\mathrm{supp}(f) = \{x \in E \mid f(x) \neq 0\}$
   2. $\mathbf{k}^{(E)} = \{f \in \mathbf{k}^E \mid \#(\mathrm{supp}(f)) < \infty\}$
   3. $\langle S|P\rangle = \sum_{x\in E} S(x)P(x),\ S \in \mathbf{k}^E,\ P \in \mathbf{k}^{(E)}$
7. Starting with a monoid $(M, ., 1_M)$ and considering $\mathbf{k}^{(M)} = \mathbf{k}[M] \subset \mathbf{k}[[M]] = \mathbf{k}^M$, we see that in order to extend the product formula

$$P.Q := \sum_{u.v=w} \langle P|u\rangle\langle Q|v\rangle\, w \tag{2}$$

it is sufficient (and necesary in general position) that the map $\star : M \times M \to M$ has finite fibers[a] (condition [D], see [4] III §2.10).

[a] Recall that a map $f:\ X \to Y$ between two sets X and Y has finite fibers if and only if for each $y \in Y$, the preimage $f^{-1}(y)$ is finite.

Which monoids are admissible for producing Magnus Groups ?/2

8. If M satisfies condition [D], we can extend the formula (2) to arbitrary $P, Q \in \mathbf{k}^M$ (as opposed to merely $P, Q \in \mathbf{k}[M]$). In this case, the $\mathbf{k}$-algebra $\left(\mathbf{k}^M, ., 1_M\right)$ is called the total algebra of M, [a] and its product is the Cauchy product between series.
9. For every $S \in \mathbf{k}^M$, the family $(\langle S|m\rangle\, m)_{m\in M}$ is summable[b]. and its sum is precisely $S = \sum_{m\in M} \langle S|m\rangle\, m$.

---

[a] See also `https://en.wikipedia.org/wiki/Total_algebra`.

[b] We say that a family $(f_i)_{i\in I}$ of elements of $\mathbf{k}^M$ is summable if for any given $m \in M$, all but finitely many $i \in I$ satisfy $\langle f_i|m\rangle = 0$. Such a summable family will always have a well-defined infinite sum $f = \sum_{m\in M} \sum_{i\in I} \langle f_i|m\rangle\, m \in \mathbf{k}^M$, whence the name "summable".

Which monoids are admissible for producing Magnus Groups ?/3

10. For example, the monoid $M = \{x^k\}_{k \in \mathbb{Z}}$, a multiplicative copy of $\mathbb{Z}$ does not satisfy condition [D].
11. Then, $\mathbf{k}[M] = \mathbf{k}[x, x^{-1}]$ is the algebra of Laurent polynomials. It admits no total algebra.
12. For this monoid, we have to impose a constraint of the support (i.e. admit only supports like $[a, +\infty[_{\mathbb{Z}}$. The resulting algebra, $\mathbf{k}[x, x^{-1}]]$ is that of Laurent series.

Which monoids are admissible for producing Magnus Groups ?/4

13. For every series $S \in \mathbf{k}[[M]]$, we set $S_+ := \sum_{m\neq 1}\langle S|m\rangle\, m$.
In order for the family $((S_+)^n)_{n\geq 0}$ to be summable, it is sufficient that the iterated multiplication map $\mu^* :\ (M_+)^* \to M$ defined by

$$\mu^*[m_1,\ldots,m_n] = m_1\cdots m_n \text{ (product within } M) \tag{3}$$

have finite fibers (where we have written the word $[m_1,\ldots,m_n] \in (M_+)^*$ as a list to avoid confusion).[a]

14. In this case the characteristic series of M (i.e. $\underline{M} = \sum_{m\in M} m = 1 + \underline{M_+}$) is invertible and its inverse is called the Möbius function $\mu :\ M \to \mathbb{Z}$. It is such that

$$\underline{M}^{-1} = 1 - \underline{M_+} + \underline{M_+}^2 - \underline{M_+}^3 - \cdots = \sum_{m\in M}\mu(m).m \tag{4}$$

[a] Furthermore, this condition is also necessary (if S_+ is generic) if $\mathbf{k} = \mathbb{Z}$. These monoids are called “locally finite” in [18].

Examples and remarks

15. Every finite monoid (and in particular finite groups) satisfies condition (D).
16. Among finite groups, only the trivial group is locally finite.
17. Many combinatorial monoids are such that $M_+ = M \setminus \{1_M\}$ is stable by products.
18. For example X^*, $X^* \otimes X^*$ and $\mathbb{N}^{(X)}$ (the free abelian monoid)
19. In the case of point 17, $S \mapsto \langle S | 1_M \rangle$ is a character of $\mathbf{k}[[M]]$ (with values in $\mathbf{k}$).
20. In the case of point 18, these monoids are locally finite, each M^{-1} is polynomial and given by, respectively

$$1 - X \ ; \ 1 - \sum_{x \in X} (x \otimes 1 + 1 \otimes x) + \sum_{x,y \in X} x \otimes y \ ; \ \prod_{x \in X} (1 - x) \tag{5}$$

where $\mathbb{N}^{(X)}$ is written multiplicatively $\{X^\alpha\}_{\alpha \in \mathbb{N}^{(X)}}$.

$\mathfrak{M}$-adic setting

21 Let M be a locally finite monoid and $\mathfrak{M} = \mathbf{k}_+[[M]] = \ker(\epsilon)$ where $\epsilon(T) = \langle T | 1_M \rangle$.

22 As M is locally finite, one has

$$\bigcap_{n \geq 0} \mathfrak{M}^n = \{0\} \tag{6}$$

where

$$\mathbf{k}[[M]] = \mathfrak{M}^0 \supset \mathfrak{M}^1 \supset \mathfrak{M}^2 \supset \cdots \supset \mathfrak{M}^n \supset \mathfrak{M}^{n+1} \supset \cdots \tag{7}$$

23 For example, with $M = X^*$, we have $\mathfrak{M}^n = \mathbf{k}_{\geq n}\langle\langle X \rangle\rangle$

24 General setting : Let R be a ring and $\mathfrak{M}$ a two-sided ideal in R. We suppose that (6) is fulfilled. With this setting, we can construct a valuation: for $m \in R$, we define

$$\varpi(m) = \sup_{\mathbb{N} \cup \{+\infty\}} \{ n \in \mathbb{N} \,|\, m \in \mathfrak{M}^n \} \tag{8}$$

$\mathfrak{M}$-adic setting/2

21 and given $0 < \rho < 1$, we define, for $r, s \in R$

$$d_\rho(r,s) = \rho^{\varpi(r-s)} \tag{9}$$

22 One can show that R, endowed with this distance function[a] is a metric topological ring. This means that the operations $(x,y) \to x+y$ and $(x,y) \to xy$ are continuous[b].

23 Great examples are $(\mathbb{Z}, p\mathbb{Z})$ (with p prime) and in the noncommutative world (with $|X| \geq 2$) $(\mathbf{k}\langle\langle X\rangle\rangle, \mathbf{k}_+\langle\langle X\rangle\rangle)$ ($\mathbf{k}_+\langle\langle X\rangle\rangle$ is maximal only iff $\mathbf{k}$ is a field)

24 We remark at once that we are not stuck to the $\mathfrak{M}$-adic setting, it suffices to have at hand a filtration i.e. a family of subgroups

$$R = \mathfrak{M}_0 \supset G_1 \supset G_2 \supset \cdots \supset G_n \supset G_{n+1} \supset \cdots \tag{10}$$

(s.t. $G_pG_q \subset G_{p+q}$) and define a distance like in (8,9).

[a] This is a distance because of the relation (6) otherwise it is a pseudometric.

[b] This definition is complete because $x \to -x$ is continuous from product.

Topological rings

25. Even worse (or better), there are examples of topological rings(which we use) where the convergence is not defined by subgroups. This is the case of $\mathcal{H}(\Omega)$ (archimedean in the sense where for all neighbourhood of zero V and $f \in \mathcal{H}(\Omega)$, there exists $n \geq 1$ such that $\frac{1}{n}.f \in V$).
26. Then we need a larger framework because $\mathbf{k}\langle\langle Y \rangle\rangle$ is not $\mathfrak{M}$-adic and $\mathcal{H}(\Omega)$ is not linearly topologizable.
27. In every topological (abelian) group $(G, +)$ the system of neighbourhoods $\mathfrak{B}(x)$ of $x \in G$, is deduced from that of zero by translation i.e.

$$\mathfrak{B}(x) = \{x + W\}_{W \in \mathfrak{B}(0)}$$

28. The following exercise is LTTR (left to the reader). For such a degree of generality, please have a look to [5, 7, 8].

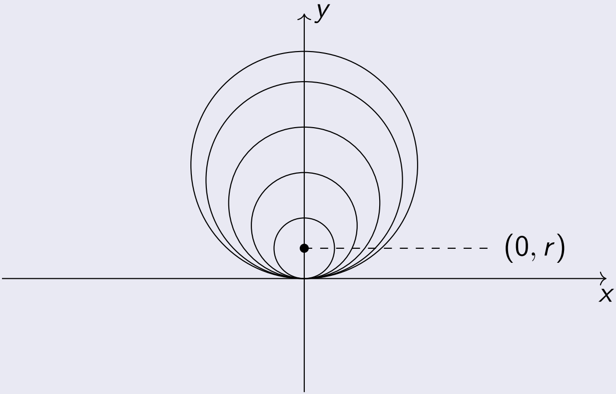
29 **Ex7. –**
Let $(R, +, \times)$ be a ring, we say that a topology τ is compatible with the ring structure of R if $(x, y) \to x + y$ and $(x, y) \to xy$ are continuous.
1) Let τ be a topology on R prove that TFAE
i) τ is compatible with the ring structure of R.
ii) a) $(x, y) \to x + y$ is continuous at $(0, 0)$
b) $(x, y) \to xy$ is continuous at $(0, 0)$.
c) For all $x_0 \in R$), $x \to x.x_0$ and $x \to x_0.x$ are continuous at 0.
Hint. – For ii.(b+c)$\Longrightarrow$ i) consider that

$$xy - uv = (x - u)(y - v) + u(y - v) + (x - u)v.$$

2) Let $(R, +)$ be an abelian group.
a) Let $\mathfrak{B} = \mathfrak{B}(0)$ be a filter basis around 0 ($0 \in V$ for all $V \in \mathfrak{B}$) such that forall $V \in \mathfrak{B}$
i) $(\exists W \in \mathfrak{B})(W + W \subset V)$.
ii) $(\exists W \in \mathfrak{B})(W \subset -V)$.
Show that the system $\mathfrak{B}(x) = \mathfrak{B} + x$ defines a topology on R.
b) Let $R = \mathbb{R}^2$ and $\mathfrak{B}$ the family of disks $(B_{(0,r)}(r))_{r>0}$ satisfies (a.i), not (a.ii) and defines no topology on R.

The family $\mathfrak{B}$.

30. The family of disks $(B_{(0,r)}(r))_{r>0}$.



Metric rings and pre-convolution.

31 **Ex8. –**
Let $(R, +, \times)$ be a ring, we say that a metric d on R is compatible with the ring structure of R if the topology induced by d is so.
Q) Construct an invariant metric d_{inv}[a] compatible with the ring structure.

[a] That is to say that, for all $x, y, z \in R$, we have $d_{inv}(x+z, y+z) = d_{inv}(x, y)$

32 If we return to MRS factorization and our last interpretation of $B = C$ as an expression of $Id_{\mathbf{k}\langle X\rangle} \in End(\mathbf{k}\langle X\rangle)$.

$$\mathcal{D}_X := \sum_{w\in X^*} w \otimes w = \sum_{w\in X^*} S_w \otimes P_w = \prod_{l\in\mathcal{L}ynX}^{\searrow} \exp(S_l \otimes P_l) \qquad (11)$$

We feel that we need a structure of ring on $End(\mathbf{k}\langle X\rangle)$ explaining the infinite product.

33. If C is a $\mathbf{k}$-coalgebra, and if A is a $\mathbf{k}$-algebra, then the $\mathbf{k}$-module $\operatorname{Hom}(C, A)$ itself becomes a $\mathbf{k}$-algebra using a multiplication operation known as convolution. We denote it by $\circledast$, and recall how it is defined: For any two $\mathbf{k}$-linear maps $f, g \in \operatorname{Hom}(C, A)$, we have

$$f \circledast g = \mu_A \circ (f \otimes g) \circ \Delta_C : C \to A.$$

The map $\eta_A \circ \epsilon_C : C \to A$ is a neutral element for this operation $\circledast$.

(Note that the operation $\circledast$ is denoted by $\star$ in [17, Definition 1.4.1].)

34. If f is a $\mathbf{k}$-linear map from a coalgebra C to an algebra A, and if $n \in \mathbb{N}$, then $f^{\circledast n}$ denotes the n-th power of f with respect to convolution (i.e., the n-th power of f in the algebra $(\operatorname{Hom}(C, A), \circledast, \eta_A \circ \epsilon_C)$).

If M is any $\mathbf{k}$-module, then the dual $\mathbf{k}$-module $\operatorname{Hom}_{\mathbf{k}}(M, \mathbf{k})$ shall be denoted by $M^\vee$. Thus, if C is a $\mathbf{k}$-coalgebra, then its dual $\mathbf{k}$-module $C^\vee = \operatorname{Hom}(C, \mathbf{k})$ becomes a $\mathbf{k}$-algebra via the convolution product $\circledast$. The unity of this $\mathbf{k}$-algebra $C^\vee$ is exactly the counit ϵ of C.

Making (combinatorial) bialgebras

Proposition

Let $\mathbf{k}$ *be a commutative ring (with unit). We suppose that the product* φ *is associative, then, on the algebra* $(\mathbf{k}\langle X\rangle, ⧢_\varphi, 1_{X^*})$*, we consider the comultiplication* Δ_{conc} *dual to the concatenation*

$$\Delta_{conc}(w) = \sum_{uv=w} u \otimes v \tag{12}$$

and the "constant term" character $\varepsilon(P) = \langle P|1_{X^*}\rangle$*.*
Then

(i) *With this setting, we have a bialgebra* [a]*.*

$$\mathcal{B}_\varphi = (\mathbf{k}\langle X\rangle, ⧢_\varphi, 1_{X^*}, \Delta_{conc}, \varepsilon) \tag{13}$$

(ii) *The bialgebra (eq. 13) is, in fact, a Hopf Algebra.*

[a] Commutative and, when $|X| \geq 2$, noncocommutative.

Dualizability

If one considers φ as defined by its structure constants

$$\varphi(x,y)=\sum_{z\in X}\gamma_{x,y}^{z}\,z$$

one sees at once that $\sqcup\!\sqcup_{\varphi}$ is dualizable within $\mathbf{k}\langle X\rangle$ iff the tensor $\gamma_{x,y}^{z}$ is locally finite in its contravariant place "z" i.e.

$$(\forall z\in X)(\#\{(x,y)\in X^2|\gamma_{x,y}^{z}\neq 0\}<+\infty)\ .$$

Remark

Shuffle, stuffle and infiltration are dualizable. The comultiplication associated with the stuffle with negative indices is not.

Dualizability/2

In the case when $\sqcup\!\sqcup_\varphi$ is dualizable, one has a comultiplication

$$\Delta_{\sqcup\!\sqcup_\varphi} : \mathbf{k}\langle X\rangle \to \mathbf{k}\langle X\rangle \otimes \mathbf{k}\langle X\rangle$$

such that, for all $u, v, w \in X^*$

$$\langle u \sqcup\!\sqcup_\varphi v | w\rangle = \langle u \otimes v | \Delta_{\sqcup\!\sqcup_\varphi}(w)\rangle \tag{14}$$

Then, the following

$$\mathcal{B}_\varphi^\vee = (\mathbf{k}\langle X\rangle, conc, 1_{X^*}, \Delta_{\sqcup\!\sqcup_\varphi}, \varepsilon) \tag{15}$$

is a bialgebra in duality with $\mathcal{B}_\varphi$ (not always a Hopf algebra although $\mathcal{B}$ was so, for example, see $\mathcal{B}$ with $\sqcup\!\sqcup_\varphi = \uparrow_q$ i.e. the q-infiltration).

The interest of these bialgebras is that they provide a host of easy-to-within-compute bialgebras with easy-to-implement-and-compute set of characters through the following proposition.

Proposition (Conc-Bialgebras)

Let $\mathbf{k}$ *be a commutative ring,* X *a set and* $\varphi(x,y)=\sum_{z\in X}\gamma_{x,y}^z\, z$ *an associative and dualizable law on* $\mathbf{k}.X$*. Let* $\sqcup\!\sqcup_\varphi$ *and* $\Delta_{\sqcup\!\sqcup_\varphi}$ *be the associated product and co-product. Then:*
i) $\mathcal{B}=(\mathbf{k}\langle X\rangle, conc, 1_{X^*}, \Delta_{\sqcup\!\sqcup_\varphi}, \epsilon)$ *is a bialgebra which, in case* $\mathbb{Q}\hookrightarrow\mathbf{k}$*, is an enveloping algebra iff* φ *is commutative and* $\Delta^+_{\sqcup\!\sqcup_\varphi}$ *nilpotent.*
ii) In the general case $S\in\mathbf{k}\langle\langle X\rangle\rangle=\mathbf{k}\langle X\rangle^\vee$ *is a character for* $\mathcal{A}=(\mathbf{k}\langle X\rangle, conc, 1_{X^*})$ *(i.e. a conc-character) iff it is of the form*

$$S=\Big(\sum_{x\in X}\alpha_x\, x\Big)^*=\sum_{n\ge 0}\Big(\sum_{x\in X}\alpha_x\, x\Big)^n \text{ and, with this notation} \tag{16}$$

$$\Big(\sum_{x\in X}\alpha_x\, x\Big)^*\sqcup\!\sqcup_\varphi\Big(\sum_{x\in X}\beta_y\, y\Big)^*=\left(\sum_{z\in X}(\alpha_z+\beta_z)\, z+\sum_{x,y\in X}\alpha_x\beta_y\,\varphi(x,y)\right)^* \tag{17}$$

GD, Darij Grinberg and Hoang Ngoc Minh *Three variations on the linear independence of grouplikes in a coalgebra*, [arXiv:2009.10970]

GD, Quoc Huan Ngô and V. Hoang Ngoc Minh, *Kleene stars of the plane, polylogarithms and symmetries*, (pp 52-72) TCS 800, 2019, pp 52-72.

Main result about independence of characters w.r.t.

Theorem (G.D., Darij Grinberg, H. N. Minh)

Let $(\mathcal{B}, ., 1_{\mathcal{B}}, \Delta, \epsilon)$ *be a* $\mathbf{k}$*-bialgebra. As usual, let* $\Delta = \Delta_{\mathcal{B}}$ *and* $\epsilon = \epsilon_{\mathcal{B}}$ *are its comultiplication and its counit.*
Let $\mathcal{B}_+ = \ker(\epsilon)$*. For each* $N \geq 0$*, let* $\mathcal{B}_+^N = \underbrace{\mathcal{B}_+ \cdot \mathcal{B}_+ \cdot \cdots \cdot \mathcal{B}_+}_{N \text{ times}}$*, where* $\mathcal{B}_+^0 = \mathcal{B}$*. Note that* $\left(\mathcal{B}_+^0, \mathcal{B}_+^1, \mathcal{B}_+^2, \ldots\right)$ *is called the* <u>*standard decreasing filtration*</u> *of* $\mathcal{B}$*.*
For each $N \geq -1$*, we define a* $\mathbf{k}$*-submodule* $\mathcal{B}_N^\vee$ *of* $\mathcal{B}^\vee$ *by*

$$\mathcal{B}_N^\vee = (\mathcal{B}_+^{N+1})^\perp = \left\{ f \in \mathcal{B}^\vee \mid f\left(\mathcal{B}_+^{N+1}\right) = 0 \right\}. \tag{18}$$

Thus, $\left(\mathcal{B}_{-1}^\vee, \mathcal{B}_0^\vee, \mathcal{B}_1^\vee, \ldots\right)$ *is an increasing filtration of* $\mathcal{B}_\infty^\vee := \bigcup_{N \geq -1} \mathcal{B}_N^\vee$ *with* $\mathcal{B}_{-1}^\vee = 0$*.*

Theorem (DGM, cont'd)

Let also $\Xi(\mathcal{B})$ be the monoid (group, if $\mathcal{B}$ is a Hopf algebra) of characters of the algebra $(\mathcal{B}, \mu_{\mathcal{B}}, 1_{\mathcal{B}})$.
Then:

(a) *We have $\mathcal{B}_p^\vee \circledast \mathcal{B}_q^\vee \subseteq \mathcal{B}_{p+q}^\vee$ for any $p, q \geq -1$ (where we set $\mathcal{B}_{-2}^\vee = 0$). Hence, $\mathcal{B}_\infty^\vee$ is a subalgebra of the convolution algebra $\mathcal{B}^\vee$.*

(b) *Assume that $\mathbf{k}$ is an integral domain. Then, the set $\Xi(\mathcal{B})^\times$ of invertible characters (i.e., of invertible elements of the monoid $\Xi(\mathcal{B})$) is left $\mathcal{B}_\infty^\vee$-linearly independent.*

Remark

The standard decreasing filtration of $\mathcal{B}$ is weakly decreasing, it can be stationary after the first step. An example can be obtained by taking the universal enveloping bialgebra of any simple Lie algebra (or, more generally, of any perfect Lie algebra); it will satisfy $\bigcap_{n\geq 0} \mathcal{B}_+^n = \mathcal{B}_+$.

Corollary

We suppose that $\mathcal{B}$ is cocommutative, and $\mathbf{k}$ *is an integral domain. Let $(g_x)_{x \in X}$ be a family of elements of $\Xi(\mathcal{B})^\times$ (the set of invertible characters of $\mathcal{B}$), and let $\varphi_X : \mathbf{k}[X] \to (\mathcal{B}^\vee, \circledast, \epsilon)$ be the* $\mathbf{k}$*-algebra morphism that sends each $x \in X$ to g_x. In order for the family $(g_x)_{x \in X}$ (of elements of the commutative ring $(\mathcal{B}^\vee, \circledast, \epsilon)$) to be algebraically independent over the subring $(\mathcal{B}^\vee_\infty, \circledast, \epsilon)$, it is necessary and sufficient that the monomial map*

$$\begin{aligned} m : \mathbb{N}^{(X)} &\to (\mathcal{B}^\vee, \circledast, \epsilon), \\ \alpha &\mapsto \varphi_X(X^\alpha) = \prod_{x \in X} g_x^{\alpha_x} \end{aligned} \tag{19}$$

(where α_x means the x-th entry of α) be injective.

Examples

Let $\mathbf{k}$ be an integral domain, and let us consider the standard bialgebra $\mathcal{B} = (\mathbf{k}[x], \Delta, \epsilon)$ For every $c \in \mathbf{k}$, there exists only one character of $\mathbf{k}[x]$ sending x to c; we will denote this character by $(c.x)^* \in \mathbf{k}[[x]]$ (motivation about this notation is Kleene star). Thus, $\Xi(\mathcal{B}) = ((c.x)^* \mid c \in \mathbf{k})$. It is easy to check that $(c_1.x)^* \sqcup\!\sqcup (c_1.x)^* = \big((c_1 + c_2).x\big)^*$ for any $c_i \in \mathbf{k}$ $({\color{red}*})$. Thus, any $c_1, c_2, \ldots, c_k \in \mathbf{k}$ and any $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{N}$ satisfy

$$\begin{aligned}
&((c_1.x)^*)^{\sqcup\!\sqcup\,\alpha_1} \sqcup\!\sqcup ((c_2.x)^*)^{\sqcup\!\sqcup\,\alpha_2} \sqcup\!\sqcup \cdots \sqcup\!\sqcup ((c_k.x)^*)^{\sqcup\!\sqcup\,\alpha_k} \\
&= \big((\alpha_1 c_1 + \alpha_2 c_2 + \cdots + \alpha_k c_k).x\big)^* . \qquad (20)
\end{aligned}$$

From $({\color{red}*})$ above, the monoid $\Xi(\mathcal{B})$ is isomorphic with the abelian group $(\mathbf{k}, +, 0)$; in particular, it is a group, so that $\Xi(\mathcal{B})^\times = \Xi(\mathcal{B})$.

Examples/2

1. Take $\mathbf{k} = \overline{\mathbb{Q}}$ (the algebraic closure of $\mathbb{Q}$) and $c_n = \sqrt{p_n} \in \mathbf{k}$, where p_n is the n-th prime number. What precedes shows that the family of series $\left((\sqrt{p_n}x)^*\right)_{n\geq 1}$ is algebraically independent over the polynomials (i.e., over $\overline{\mathbb{Q}}[x]$) within the commutative $\overline{\mathbb{Q}}$-algebra $\left(\overline{\mathbb{Q}}[[x]], ш, 1\right)$. This example can be double-checked using partial fractions decompositions as, in fact, $(\sqrt{p_n}x)^* = \dfrac{1}{1-\sqrt{p_n}x}$ (this time, the inverse is taken within the ordinary product in $\mathbf{k}[[x]]$) and

$$\left(\frac{1}{1-\sqrt{p_n}x}\right)^{ш\, n} = \frac{1}{1-n\sqrt{p_n}x}.$$

2. This example is even more telling if you use the continuous transform $x^n \mapsto x^{ш\, n}$

$$\left(\overline{\mathbb{Q}}[[x]], \mathrm{conc}, 1\right) \to \left(\overline{\mathbb{Q}}[[x]], ш, 1\right)$$

and its inverse, the Borel transform $\mathbf{B}:\ x^n \mapsto \frac{x^n}{n!}$. We have

$$\mathbf{B}\left[(\sqrt{p_n}x)^*\right] = e^{\sqrt{p_n}x}$$

Examples/3

The preceding example can be generalized as follows: Let $\mathbf{k}$ still be an integral domain; let V be a $\mathbf{k}$-module, and let $\mathcal{B} = \big(T(V), \mathrm{conc}, 1_{T(V)}, \Delta_{\boxtimes}, \epsilon\big)$ be the standard tensor conc-bialgebra[a] For every linear form $\varphi \in V^{\vee}$, there is an unique character φ^* of $\big(T(V), \mathrm{conc}, 1_{T(V)}\big)$ such that all $u \in V$ satisfy

$$\langle \varphi^* | u \rangle = \langle \varphi | u \rangle. \tag{21}$$

Again, it is easy to check[b] that $(\varphi_1)^* \sqcup\!\sqcup (\varphi_2)^* = \big(\varphi_1 + \varphi_2\big)^*$ for any $\varphi_1, \varphi_2 \in V^{\vee}$, because both sides are characters of $\big(T(V), \mathrm{conc}, 1_{T(V)}\big)$ so that the equality has only to be checked on V.

---

[a] The one defined by

$$\Delta_{\boxtimes}(1) = 1 \otimes 1 \text{ and } \Delta_{\boxtimes}(u) = u \otimes 1 + 1 \otimes u \ ; \ \epsilon(u) = 0 \text{ for all } u \in V.$$

[b] For this bialgebra $\sqcup\!\sqcup$ stands for $\circledast$ on the space $\mathrm{Hom}(\mathcal{B}, \mathbf{k})$.

Examples/4

Again, from this, any $\varphi_1, \varphi_2, \ldots, \varphi_k \in V^\vee$ and any $\alpha_1, \alpha_2, \ldots, \alpha_k \in \mathbb{N}$ satisfy

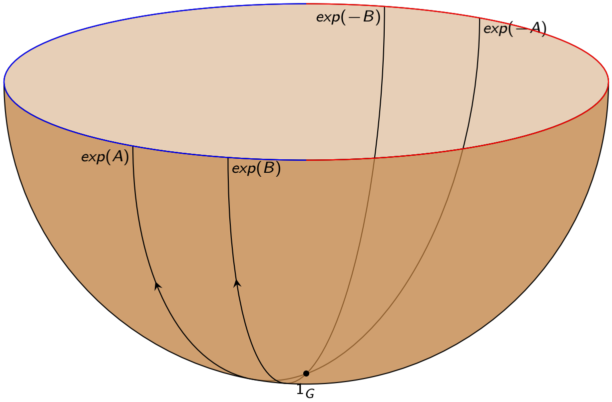
$$\begin{aligned} &\left((\varphi_1)^*\right)^{\sqcup\!\sqcup\, \alpha_1} \sqcup\!\sqcup \left((\varphi_2)^*\right)^{\sqcup\!\sqcup\, \alpha_2} \sqcup\!\sqcup \cdots \sqcup\!\sqcup \left((\varphi_k)^*\right)^{\sqcup\!\sqcup\, \alpha_k} \\ &= \left(\alpha_1\varphi_1 + \alpha_2\varphi_2 + \cdots + \alpha_k\varphi_k\right)^* . \end{aligned} \tag{22}$$

The decreasing filtration of $\mathcal{B}$ is given by $\mathcal{B}_+^n = \bigoplus_{k\geq n} T_k(V)$ (the ideal of tensors of degree $\geq n$) and the reader may check easily that, in this case, $\mathcal{B}_\infty^\vee$ is the shuffle algebra of finitely supported linear forms i.e., for each $\Phi \in \mathcal{B}^\vee$, we have the equivalence

$$\Phi \in \mathcal{B}_\infty^\vee \iff (\exists N \in \mathbb{N})(\forall k \geq N)(\Phi(T_k(V)) = \{0\}).$$

Then, Corollary above shows that $(\varphi_i^*)_{i\in I}$ are $\mathcal{B}_\infty^\vee$-algebraically independent within $(T(V)^\vee, \sqcup\!\sqcup, \epsilon)$ iff the corresponding monomial map is injective, and (22) shows that it is so iff the family $(\varphi_i)_{i\in I}$ of linear forms is $\mathbb{Z}$-linearly independent in $V^\vee$.

Magnus and Hausdorff groups



The Magnus group is the set of series with constant term 1_{X^*}, the Hausdorff (sub)-group, is the group of group-like series for $\Delta_{⊔⊔}$. These are also Lie exponentials (here A, B are Lie series and $exp(A)exp(B) = exp(H(A, B))$).

Hausdorff group of the stuffle Hopf algebra.

With $Y = \{y_i\}_{i\geq 1}$ and

$$\Delta_{\,⧢}(y_k) = y_k \otimes 1 + 1 \otimes y_k + \sum_{i+j=k} y_i \otimes y_j$$

the bialgebra $\mathcal{B} = (\mathbf{k}\langle X\rangle, \mathtt{conc}, 1_{X^*}, \Delta_{\,⧢}, \epsilon)$ is an enveloping algebra (it is cocommutative, connex and graded by the weight function given by $||y_{i_1}y_{i_2}\cdots y_{i_k}|| = \sum_{s=1}^{k} i_s$ on a word $w = y_{i_1}y_{i_2}\cdots y_{i_k}$).
With $\varphi(y_i, y_j) = y_{i+j}$, (eq.17) gives

$$(\sum_{i\geq 1}\alpha_i\, y_i)^*_{\,⧢}(\sum_{j\geq 1}\beta_j\, y_j)^* = (\sum_{i\geq 1}\alpha_i\, y_i + \sum_{j\geq 1}\beta_j\, y_j + \sum_{i,j\geq 1}\alpha_i\beta_j\, y_{i+j})^* \qquad (23)$$

This formula suggests us to code, in an umbral style, $\sum_{k\geq 1}\alpha_k\, y_k$ by the series $\sum_{k\geq 1}\alpha_k\, x^k \in \mathbf{k}_+[[x]]$. Indeed, we get the following proposition whose first part, characteristic-freely describes the group of characters $\Xi(\mathcal{B})$ and its law and the second part, about the exp-log correspondence, requires $\mathbf{k}$ to be $\mathbb{Q}$-algebra.

Proposition

Let π_Y^{Umbra} *be the linear isomorphism* $\mathbf{k}_+[[x]] \to \widehat{\mathbf{k}.Y}$ *defined by*

$$\sum_{n\geq 1} \alpha_n\, x^n \mapsto \sum_{k\geq 1} \alpha_k\, y_k \tag{24}$$

Then

1. *One has, for* $S, T \in \mathbf{k}_+[[x]]$,

$$(\pi_Y^{Umbra}(S))^* \uplus (\pi_Y^{Umbra}(T))^* = (\pi_Y^{Umbra}((1+S)(1+T)-1))^* \tag{25}$$

2. *From now on* $\mathbf{k}$ *is supposed to be a* $\mathbb{Q}$*-algebra.*
   For $t \in \mathbf{k}$ *and* $T \in \mathbf{k}_+[[x]]$, *the family* $(\frac{(t.T)^n}{n!})_{n\geq 0}$ *is summable and one sets*

$$G(t) = (\pi_Y^{Umbra}(e^{t.T} - 1))^* \tag{26}$$

Proposition (Cont'd)

3. *The parametric character G fulfills the "stuffle one-parameter group" property i.e. for $t_1, t_2 \in \mathbf{k}$, we have*

$$G(t_1 + t_2) = G(t_1) \uplus G(t_2) \; ; \; G(0) = 1_{Y^*} \tag{27}$$

4. *We have*

$$G(t) = exp_{\uplus}\left(t.\pi_Y^{Umbra}(T)\right) \tag{28}$$

5. *In particular, calling π_x^{Umbra} the inverse of π_Y^{Umbra} we get, for $P^* \in \Xi(\mathcal{B})$ (in other words $P \in \widehat{\mathbf{k}.Y}$),*

$$\log_{\uplus}(P^*) = \pi_Y^{Umbra}(\log(1 + \pi_x^{Umbra}(P))) \tag{29}$$

Proof (Sketch)

i) We have

$$\pi_Y^{Umbra}(S) = \sum_{i\geq 1}\langle S|x^i\rangle y_i \quad \pi_Y^{Umbra}(T) = \sum_{j\geq 1}\langle T|x^j\rangle y_j$$

and then

$$(\pi_Y^{Umbra}(S))^* \uplus (\pi_Y^{Umbra}(T))^* = (\sum_{i\geq 1}\langle S|x^i\rangle y_i)^* \uplus (\sum_{j\geq 1}\langle T|x^j\rangle y_j) =$$
$$\Big(\sum_{i\geq 1}\langle S|x^i\rangle y_i) + \sum_{j\geq 1}\langle T|x^j\rangle y_j + \sum_{i,j\geq 1}\langle S|x^i\rangle\langle T|x^j\rangle y_{i+j}\Big)^* =$$
$$(\pi_Y^{Umbra}(S+T+ST))^* = (\pi_Y^{Umbra}((1+S)(1+T)-1))^*$$

ii.1) The one parameter group property is a consequence of (25) applied to the series $e^{t_i.T}-1,\ i=1,2.$

Proof (Sketch)/2

ii.2) Property 27 holds for every $\mathbb{Q}$-algebra, in particular in $\mathbf{k}_1 = \mathbf{k}[t]$ and $\mathbf{k}_1\langle\langle Y \rangle\rangle$ is endowed with the structure of a differential ring by term-by-term derivations (see [**?**] for formal details). We can write $G(t) = 1 + t.G_1 + t^2.G_2(t)$ (where $G_1 = \pi_Y^{Umbra}(T)$ is independent from t) and from 27, we have

$$G'(t) = G_1.G(t)\ ;\ \ G(0) = 1_{Y^*} \tag{30}$$

but $H(t) = exp_{\uplus}(t.G_1)$ satisfies 30 whence the equality.
ii.3) At $t = 1$, we have $\exp_{\uplus}(\pi_Y^{Umbra}(T)) = (\pi_Y^{Umbra}(e^T - 1))^*$ hence, with $P = \pi_Y^{Umbra}(e^T - 1)$ (take $T := \log(\pi_x^{Umbra}(P) + 1)$)

$$\pi_Y^{Umbra}(T) = \log_{\uplus}(P^*) \qquad [\mathrm{QED}] \tag{31}$$

Application of (29)

$$(ty_k)^* = \exp_{\uplus}\left(\sum_{n\geq 1} \frac{(-1)^{n-1} t^n y_{nk}}{n}\right) \tag{32}$$

Conclusion(s): More applications and perspectives.

We have seen

1. Star of the plane property (see [13]) holds for non-commutative valued (as matrix-valued) characters.
2. Combinatorial study of other $\sqcup\!\sqcup_{\varphi}$ one-parameter groups and evolution equations in convolution algebras.
3. Factorisation of $\mathcal{A}$-valued characters ($\mathcal{A}$ $\mathbf{k}$-CAAU).
   For example, with

$$\mathcal{B} = (\mathbf{k}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*}, \Delta_{conc}, \epsilon)\,,\ \mathcal{A} = (\mathbf{k}\langle X\rangle, \sqcup\!\sqcup, 1_{X^*})\,,\ \chi = Id$$

(χ is a shuffle character) one has (MRS factorisation)

$$\Gamma(\chi) = \sum_{w\in X^*} Id(w)\otimes w = \sum_{w\in X^*} S_w\otimes P_w = \prod_{l\in\mathcal{L}ynX}^{\searrow} \exp(S_l\otimes P_l) \tag{33}$$

Conclusion(s): More applications and perspectives./2

4. Deformed version of factorisation above for $\sqcup\!\sqcup_{\varphi}$ (with φ associative, commutative, dualisable and moderate). With

$$\mathcal{B} = \left(\mathbf{k}\langle X\rangle, \sqcup\!\sqcup_{\varphi}, 1_{X^*}, \Delta_{conc}, \epsilon\right), \ \mathcal{A} = \left(\mathbf{k}\langle X\rangle, \sqcup\!\sqcup_{\varphi}, 1_{X^*}\right), \ \chi = Id$$

(χ is a shuffle character) one has

$$\Gamma(\chi) = \sum_{w\in X^*} Id(w)\otimes w = \sum_{w\in X^*} \Sigma_w \otimes \Pi_w = \prod_{l\in\mathcal{L}ynX}^{\searrow} \exp(\Sigma_l \otimes \Pi_l) \tag{34}$$

5. Holds for all enveloping algebras which are free as $\mathbf{k}$-modules (with $\mathbb{Q}\hookrightarrow\mathbf{k}$). This could help to the combinatorial study of the group of characters of enveloping algebras of Lie algebras like KZ[a]-Lie algebras and other ones, or deformed.

[a] Knizhnik–Zamolodchikov.

Conclusion(s): Main theorem.

Theorem, [14]

Let $\mathbf{k}$ be a $\mathbb{Q}$-algebra and $\mathfrak{g}$ be a Lie algebra which is free as a $\mathbf{k}$-module. Let us fix an ordered basis $B = (b_i)_{i \in I}$ (where the ground set $(I, <)$ is totally ordered) of $\mathfrak{g}$. To construct the associated PBW basis of $\mathcal{U} = \mathcal{U}(\mathfrak{g})$, we use the following multiindex notation. For every $\alpha \in \mathbb{N}^{(I)}$, we set

$$B^\alpha = b_{i_1}^{\alpha(i_1)} \cdots b_{i_n}^{\alpha(i_n)} \in \mathcal{U} \tag{35}$$

where $\{i_1, \cdots, i_n\} \supset \mathrm{supp}(\alpha)$ (and $i_1 < \cdots < i_n$).
Consider the linear coordinate forms $B_\beta \in \mathcal{U}^\vee$ defined by

$$\langle B_\beta | B^\alpha \rangle = \delta_{\alpha,\beta}. \tag{36}$$

We will also use the elementary multiindices $e_i \in \mathbb{N}^{(I)}$ defined for all $i \in I$ by $e_i(j) = \delta_{i,j}$.

Conclusion(s): Main theorem/2

Theorem cont'd

Then:[a]

1. We have
$$B_\alpha \circledast B_\beta = \frac{(\alpha+\beta)\,!}{\alpha\,!\,\beta\,!} B_{\alpha+\beta} \tag{37}$$
and
$$B_{\alpha(i_1)e_{i_1}+\cdots+\alpha(i_k)e_{i_k}} = \frac{B_{e_{i_1}}^{\circledast\alpha(i_1)} \circledast \cdots \circledast B_{e_{i_k}}^{\circledast\alpha(i_k)}}{\alpha(i_1)\,!\cdots\alpha(i_k)\,!}. \tag{38}$$
2. The following infinite product identity holds:
$$Id_{\mathcal{U}} = \circledast_{\overrightarrow{i\in I}}\ e_{\circledast}^{Im(B_{e_i}\otimes B^{e_i})} = \prod_{i\in I}^{\rightarrow}\ e_{\circledast}^{Im(B_{e_i}\otimes B^{e_i})} \tag{39}$$
within $End(\mathcal{U})$.

[a] We use the notation $\alpha!$ for $\alpha \in \mathbb{N}^{(I)}$; this is the product $\alpha! = \prod_{i\in I} \alpha_i!$.

THANK YOU FOR YOUR ATTENTION !

By the way, below the bibliography cited and some more running titles.

[1] Eiichi Abe, *Hopf algebras*, Cambridge Tracts in Mathematics **74**. Cambridge University Press, Cambridge-New York, 1980.

[2] J. W. Addison and S. C. Kleene *A note on function quantification*, Proceedings of the American Mathematical Society, **8** (1957)

[3] Van Chien Bui, Gérard H.E. Duchamp, Quoc Huan Ngô, Vincel Hoang Ngoc Minh and Christophe Tollu (Pure) Transcendence Bases in φ-Deformed Shuffle Bialgebras (22 pp.), 74 ème Séminaire Lotharingien de Combinatoire (published oct. 2018).

[4] N. Bourbaki, *Algebra I (Chapters 1-3)*, Springer 1989.

[5] N. Bourbaki, *General Topology, Part 1*, Springer-Verlag Berlin (1999).

[6] N. Bourbaki, *General Topology, Part 2*, Springer-Verlag Berlin (1989).

[7] Topological Ring in Wikipedia, `https://en.wikipedia.org/wiki/Topological_ring`

[8] Topological Ring in nLab,
`https://ncatlab.org/nlab/show/topological+ring`

[9] V. Drinfel'd, *On quasitriangular quasi-hopf algebra and a group closely connected with Gal($\bar{\mathbb{Q}}/\mathbb{Q}$)*, Leningrad Math. J., 4, 829-860, 1991.

[10] G.H.E. Duchamp, J.Y. Enjalbert, Hoang Ngoc Minh, C. Tollu, *The mechanics of shuffle products and their siblings*, Discrete Mathematics 340(9): 2286-2300 (2017).

[11] G.H.E. Duchamp, D. Grinberg, V. Hoang Ngoc Minh, *Three variations on the linear independence of grouplikes in a coalgebra*, arXiv:2009.10970 [math.QA]

[12] M. Deneufchâtel, G.H.E. Duchamp, V. Hoang Ngoc Minh and A. I. Solomon, *Independence of Hyperlogarithms over Function Fields via Algebraic Combinatorics*, 4th International Conference on Algebraic Informatics, Linz (2011). Proceedings, Lecture Notes in Computer Science, 6742, Springer.

[13] GD, Quoc Huan Ngô and Vincel Hoang Ngoc Minh, *Kleene stars of the plane, polylogarithms and symmetries*, (pp 52-72) TCS 800, 2019, pp 52-72.

[14] GD, Darij Grinberg and Vincel Hoang Ngoc Minh, *Kleene stars in shuffle algebras*, (in preparation).

[15] G. H. E Duchamp (LIPN), N. Gargava (EPFL), Hoang Ngoc Minh (LIPN), P. Simonnet (SPE), *A localized version of the basic triangle theorem.*, r arXiv:1908.03327.

[16] R.L. Graham, D.E. Knuth, O. Patashnik, *Concrete Mathematics: A Foundation for Computer Science (2nd Edition)*, Addison-Wesley (672 pages).

[17] Darij Grinberg, Victor Reiner, *Hopf algebras in Combinatorics*, version of 27 July 2020, arXiv1409.8356v7.
See also `http://www.cip.ifi.lmu.de/~grinberg/algebra/HopfComb-sols.pdf` for a version that gets updated.

[18] S. Eilenberg, *Automata, languages and machines, vol A*. Acad. Press, New-York, 1974.

[19] J. Esmonde, M. R. Murty, *Problems in algebraic number theory*, 2nd ed. Springer Science+Business Media, Inc. (2005).

[20] Yiannis N. Moschovakis, *Descriptive Set Theory*, Second Edition, Mathematical Surveys and Monographs, **155** (2009)

[21] Topology on the set of analytic functions `https://mathoverflow.net/questions/140441`

[22] P. Montel.– *Leçons sur les familles normales de fonctions analytiques et leurs applications*, Gauthier-Villars (1927)

[23] Van der Put, Marius, Singer, Michael F., *Galois Theory of Linear Differential Equations*, Springer; (2002)

[24] M. van der Put, *Recent work on differential Galois theory*, Séminaire N. Bourbaki, 1997-1998, exp. n o 849, p. 341-367.

[25] Ore condition
`https://en.wikipedia.org/wiki/Ore_condition`

[26] Ore localization
`https://ncatlab.org/nlab/show/Ore+domain`

[27] Constants in localizations.
`https://math.stackexchange.com/questions/2051634`

[28] Initial topology
`https://en.wikipedia.org/wiki/Initial_topology`

[29] David E. Radford, *Hopf algebras*, Series on Knots and Everything **49**. World Scientific, 2012.

[30] Christophe Reutenauer, *Free Lie Algebras*, Université du Québec a Montréal, Clarendon Press, Oxford (1993)

[31] Riemann Series Theorem
`https://en.wikipedia.org/wiki/Riemann_series_theorem`

[32] François Treves, Topological Spaces, Distributions and Kernels, Dover (2007)

[33] G. H. E. Duchamp and C. Tollu, *Sweedler's duals and Schützenberger's calculus*, In K. Ebrahimi-Fard, M. Marcolli and W. van Suijlekom (eds), Combinatorics and Physics, p. 67 - 78, Amer. Math. Soc. (Contemporary Mathematics, vol. 539), 2011. arXiv:0712.0125v3 [math.CO]

[34] Universal enveloping algebra.
`https://en.wikipedia.org/wiki/Universal_enveloping_algebra`

[35] Moss E. Sweedler, *Hopf algebras*, W.A. Benjamin, New York, 1969.